Classifications of Symmetric Normal Form Games

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Normal Form Games

Example (Two-Player Game)

	c	d
a	$3, 3$	$1, 4$
b	$4, 1$	$2, 2$

How is such a game played?
What does a normal form game consist of?

Notation

- $N = \{1, 2\}$; (set of players)
- $A_1 = \{a, b\}$, $A_2 = \{c, d\}$; (strategy sets)
- $A = A_1 \times A_2 = \{(a, c), (a, d), (b, c), (b, d)\}$; (strategy profiles)
- $u_1, u_2 : A \to \mathbb{R}$; (payoff/utility functions)
- $u_2(b, c) = 1$.

Normal Form Games

Definition

A **normal form game** Γ consists of a (finite) set N of at least two players, and for each player $i \in N$:

- A non-empty (finite) set of **strategies** A_i; and
- A **payoff/utility** function $u_i : A \to \mathbb{R}$ where $A = \times_{i \in N} A_i$ is the set of **strategy profiles**.

Normal Form Games

Example (Three-Player Game)

	e	f
c	$1,1,1$	$2,2,3$
d	$2,3,2$	$5,4,4$

$(a,,)$

	e	f
c	$3,2,2$	$4,5,4$
d	$4,4,5$	$6,6,6$

$(b,,)$

- $N = \{1,2,3\}$;
- $A_1 = \{a,b\}$, $A_2 = \{c,d\}$, $A_3 = \{e,f\}$;
- $A = \{(a,c,e),(a,c,f),(a,d,e),(a,d,f),(b,c,e),(b,c,f),(b,d,e),(b,d,f)\}$;
- $u_3(b,d,e) = 5$.

Player Permutations S_N Acting on Strategy Profiles A

Suppose each player has the same strategy set.

$$\text{Eg. } A_1 = A_2 = A_3 = \{a, b\}.$$

Let $\pi \in S_N$ be a permutation of the players.

Proposition

The player permutations act on the left of strategy profiles via

$$\pi(s_1, \ldots, s_n) = (s_{\pi^{-1}(1)}, \ldots, s_{\pi^{-1}(n)}).$$

Example

Take $\pi = (123) \in S_3$ and $(s_1, s_2, s_3) \in A$.

$$\pi(s_1, s_2, s_3) = (s_{\pi^{-1}(1)}, s_{\pi^{-1}(2)}, s_{\pi^{-1}(3)}) = (s_3, s_1, s_2)$$

Eg. $\pi(a, b, a) = (a, a, b)$

Game Invariants

Definition (von Neumann)

$\pi \in S_N$ is an **invariant** of a game Γ if for each player $i \in N$ and strategy profile $s \in A$, $u_i(s) = u_{\pi(i)}(\pi(s))$.

Invariants give us a notion of players being indifferent between current positions and an alternative arrangement of positions.

Example

	a	b
a	1, 1, 1	2, 2, 3
b	2, 3, 2	5, 4, 4

$(a, ,)$

	a	b
a	3, 2, 2	4, 5, 4
b	4, 4, 5	6, 6, 6

$(b, ,)$

- (123) and (23) are invariants of Γ;

Eg. Let $\pi = (123)$, then $u_2(a, b, a) = u_{\pi(2)}(\pi(a, b, a)) = u_3(a, a, b) = 3$.

- $\langle(123), (23)\rangle = S_3$ (invariants of Γ).

Label-Dependent Notions of Symmetry

Definition

Γ is:

- **fully symmetric** (vNM) if it is invariant under S_N; and
- **standard symmetric** (Stein?) if it is invariant under a transitive subgroup of S_N.

Example (Standard Symmetric Three-Player Game)

	a	b
a	$1, 1, 1$	$2, 3, 4$
b	$3, 4, 2$	$5, 6, 7$

$(a, ,)$

	a	b
a	$4, 2, 3$	$7, 5, 6$
b	$6, 7, 5$	$8, 8, 8$

$(b, ,)$

- Γ is invariant under (123) and not invariant under (23);
- $\langle(123)\rangle = \{e, (123), (132)\}$ is a transitive subgroup of S_3;

Note: Must have $u_i(a, a, a) = u_j(a, a, a)$ for all $i, j \in N$ etc.

Are We There Yet?

Questions

- What if players have different strategy sets?
- Have we fully captured fairness? No

Example (Matching Pennies)

	H	T
H	$1, -1$	$-1, 1$
T	$-1, 1$	$1, -1$

Game Bijections

Definition (Nash)

A **bijection** from Γ to itself consists of a player permutation $\pi \in S_N$ and for each player $i \in N$, a strategy set bijection $\tau_i : A_i \to A_{\pi(i)}$.
Notation: $\mathrm{Bij}(\Gamma)$ denotes the game bijections from Γ to itself.

Example

$$g = \left((123); \left(\begin{smallmatrix} a & b \\ d & c \end{smallmatrix}\right), \left(\begin{smallmatrix} c & d \\ e & f \end{smallmatrix}\right), \left(\begin{smallmatrix} e & f \\ a & b \end{smallmatrix}\right)\right)$$

Note: $\mathrm{Bij}(\Gamma) \cong (S_m \,\mathrm{Wr}\, S_n)$.

Proposition

Game bijections act on the left of players and strategy profiles.

Example

$$g(2) = 3 \text{ and } g(b, d, e) = (a, c, f)$$

Game Bijections

Definition

Let $G \leq \text{Bij}(\Gamma)$. The **stabiliser of player** $i \in N$ is the subgroup $G_i = \{g \in G : g(i) = i\} \leq G$.

Properties (Stein)

We say that G is:

- **player transitive** if for each $i, j \in N$ there exists $g \in G$ such that $g(i) = j$;
- **player** n**-transitive** if for each $\pi \in S_N$ there exists $g \in G$ such that $g(i) = \pi(i)$ for all $i \in N$; and
- **strategy trivial** if for each $g \in G_i$, $g(s_i) = s_i$ for all $s_i \in A_i$.

Theorem (Stein)

Strategy trivial subgroups act on strategy profiles equivalently to permutations for some relabelling of the strategies.

Automorphism Group

Definition (Nash)

An **automorphism** of Γ is an invariant bijection $g \in \text{Bij}(\Gamma)$.

$$\text{Ie. } u_i(s) = u_{g(i)}(g(s)) \text{ for all } i \in N,\ s \in A.$$

The automorphisms of Γ form a group which we denote as $\text{Aut}(\Gamma)$.

Example (Matching Pennies)

	H	T
H	$1, -1$	$-1, 1$
T	$-1, 1$	$1, -1$

$$\begin{aligned}\text{Aut}(\Gamma) = \{ & \left(e; \left(\begin{smallmatrix} H & T \\ H & T \end{smallmatrix}\right), \left(\begin{smallmatrix} H & T \\ H & T \end{smallmatrix}\right)\right), \left(e; \left(\begin{smallmatrix} H & T \\ T & H \end{smallmatrix}\right), \left(\begin{smallmatrix} H & T \\ T & H \end{smallmatrix}\right)\right), \\ & \left((12); \left(\begin{smallmatrix} H & T \\ H & T \end{smallmatrix}\right), \left(\begin{smallmatrix} H & T \\ T & H \end{smallmatrix}\right)\right), \left((12); \left(\begin{smallmatrix} H & T \\ T & H \end{smallmatrix}\right), \left(\begin{smallmatrix} H & T \\ H & T \end{smallmatrix}\right)\right)\}\end{aligned}$$

$\text{Aut}(\Gamma)$ is player n-transitive, is not strategy trivial and contains no proper transitive subgroups.

Label-Independent Notions of Symmetry

Corollary (Stein)

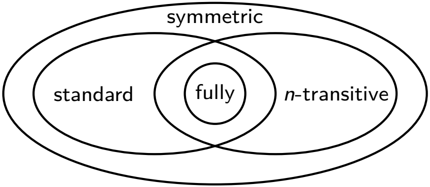
The following conditions are equivalent:

- there exists standard symmetric Γ' such that $\Gamma' \cong \Gamma$;
- $\mathrm{Aut}(\Gamma)$ has a player transitive and strategy trivial subgroup.

Definition

Γ is:

- **symmetric** if $\mathrm{Aut}(\Gamma)$ is player transitive; and
- *n*-**transitive** if $\mathrm{Aut}(\Gamma)$ is player n-transitive.



Parameterised Games

Definition

Let $G \subseteq \text{Bij}(\Gamma)$. We construct the **parameterised game** $\Gamma(G)$ **of** G by assigning a parameter to each orbit in $(N \times A)/\langle G \rangle$.

Example

$g = ((12); \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \left(\begin{smallmatrix} c & d \\ a & b \end{smallmatrix}\right))$ requires that we have,

$$u_1(a,c) = u_2(a,c) = \alpha \qquad u_1(a,d) = u_2(b,c) = \gamma$$
$$u_1(b,c) = u_2(a,d) = \beta \qquad u_1(b,d) = u_2(b,d) = \delta$$

	c	d
a	α, α	γ, β
b	β, γ	δ, δ

Note: $\langle G \rangle$ can be a proper subgroup of $\text{Aut}(\Gamma(G))$.

Parameterised Games

Example (n-Transitive Standard Non-Fully Symmetric Game)

	e	f
c	α, α, α	β, γ, δ
d	γ, δ, β	δ, γ, β

$(a, ,)$

	e	f
c	δ, β, γ	β, δ, γ
d	γ, β, δ	α, α, α

$(b, ,)$

$$G = \{((123); \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \left(\begin{smallmatrix} c & d \\ e & f \end{smallmatrix}\right), \left(\begin{smallmatrix} e & f \\ a & b \end{smallmatrix}\right)), ((12); \left(\begin{smallmatrix} a & b \\ d & c \end{smallmatrix}\right), \left(\begin{smallmatrix} c & d \\ b & a \end{smallmatrix}\right), \left(\begin{smallmatrix} e & f \\ f & e \end{smallmatrix}\right))\}$$

- $\langle G \rangle$ is player n-transitive;
- $\langle ((123); \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \left(\begin{smallmatrix} c & d \\ e & f \end{smallmatrix}\right), \left(\begin{smallmatrix} e & f \\ a & b \end{smallmatrix}\right)) \rangle$ is transitive and strategy trivial;
- $((12); \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \left(\begin{smallmatrix} c & d \\ a & b \end{smallmatrix}\right), \left(\begin{smallmatrix} e & f \\ e & f \end{smallmatrix}\right)) \notin \mathrm{Aut}(\Gamma(G))$.

Parameterised Games

Example (Only-Transitive Non-Standard Symmetric Game)

	g	h
e	$\alpha,\alpha,\beta,\beta$	$\gamma,\delta,\delta,\gamma$
f	$\delta,\gamma,\gamma,\delta$	$\beta,\beta,\alpha,\alpha$

$(a, c, ,)$

	g	h
e	$\gamma,\delta,\delta,\gamma$	$\alpha,\alpha,\beta,\beta$
f	$\beta,\beta,\alpha,\alpha$	$\delta,\gamma,\gamma,\delta$

$(a, d, ,)$

	g	h
e	$\delta,\gamma,\gamma,\delta$	$\beta,\beta,\alpha,\alpha$
f	$\alpha,\alpha,\beta,\beta$	$\gamma,\delta,\delta,\gamma$

$(b, c, ,)$

	g	h
e	$\beta,\beta,\alpha,\alpha$	$\delta,\gamma,\gamma,\delta$
f	$\gamma,\delta,\delta,\gamma$	$\alpha,\alpha,\beta,\beta$

$(b, d, ,)$

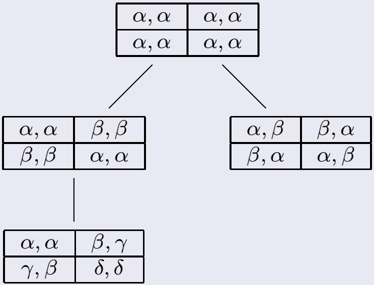
$$
\begin{aligned}
G = \{ &\left((12)\circ(34); \left(\begin{smallmatrix} a & b \\ d & c \end{smallmatrix}\right), \left(\begin{smallmatrix} c & d \\ a & b \end{smallmatrix}\right), \left(\begin{smallmatrix} e & f \\ h & g \end{smallmatrix}\right), \left(\begin{smallmatrix} g & h \\ e & f \end{smallmatrix}\right)\right), \\
&\left((13)\circ(24); \left(\begin{smallmatrix} a & b \\ f & e \end{smallmatrix}\right), \left(\begin{smallmatrix} c & d \\ h & g \end{smallmatrix}\right), \left(\begin{smallmatrix} e & f \\ a & b \end{smallmatrix}\right), \left(\begin{smallmatrix} g & h \\ c & d \end{smallmatrix}\right)\right), \\
&\left((14)\circ(23); \left(\begin{smallmatrix} a & b \\ h & g \end{smallmatrix}\right), \left(\begin{smallmatrix} c & d \\ f & e \end{smallmatrix}\right), \left(\begin{smallmatrix} e & f \\ c & d \end{smallmatrix}\right), \left(\begin{smallmatrix} g & h \\ a & b \end{smallmatrix}\right)\right)\}
\end{aligned}
$$

Partially Ordering Parameterised Games

Definition

Define $\leq$ on parameterised games as follows: $\Gamma(G) \leq \Gamma(G')$ when given a set of parameters for $\Gamma(G')$ there exists a set of parameters for $\Gamma(G)$ such that $\Gamma(G) \cong \Gamma(G')$.

Example (Symmetric 2-Player 2-Strategy Games up to Isomorphism)



Partially Ordering Parameterised Games

Example (Symmetric 3-Player 2-Strategy Games up to Isomorphism)

α, α, α	α, α, α
α, α, α	α, α, α
α, α, α	α, α, α
α, α, α	α, α, α

α, α, α	β, β, β
β, β, β	α, α, α
β, β, β	α, α, α
α, α, α	β, β, β

α, α, α	β, β, δ
β, δ, β	δ, β, β
δ, β, β	β, δ, β
β, β, δ	α, α, α

α, α, α	β, β, δ
β, δ, β	σ, ρ, ρ
δ, β, β	ρ, σ, ρ
ρ, ρ, σ	ω, ω, ω

α, α, α	β, γ, δ
γ, δ, β	δ, γ, β
δ, β, γ	β, δ, γ
γ, β, δ	α, α, α

α, α, α	β, γ, δ
γ, δ, β	δ, β, γ
δ, β, γ	γ, δ, β
β, γ, δ	α, α, α

α, α, α	β, γ, δ
γ, δ, β	σ, ρ, τ
δ, β, γ	τ, σ, ρ
ρ, τ, σ	ω, ω, ω

Questions?